3401. A hiker walks due north from camp at 3 mph. At the same time, a bear is approaching the camp from the west, walking at 5 mph. The bear arrives at the camp 6 minutes after the hiker leaves.

(a) Defining $t = 0$ as the time at which the hiker leaves, show that the bear's position can be expressed as $\mathbf{r}_b = (5t - 1/2)\mathbf{i}$, and find a similar expression for the hiker's position.

(b) Find S, the squared distance between the bear and the hiker, in terms of t.

(c) Hence, show that bear and hiker are never within a quarter of a mile of each other.

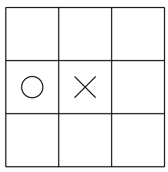
3402. A function f is defined over the reals, and has range $[a, b]$, where $a < 0 < b$ and $|a| < |b|$. Give the range of each of the following:

(a) $x \mapsto \big(\mathrm{f}(x)\big)^2$,

(b) $x \mapsto \big(\mathrm{f}(x)\big)^3$,

(c) $x \mapsto \big(\mathrm{f}(x)\big)^4$.

3403. In the game of tic-tac-toe, two players alternately place ○ and × in a grid. A player who places three in a row wins. In the game below, two moves have been played, with × to move next.



Show that, if × plays logically, then ○ has lost.

3404. Use integration by substitution to find

$$\int \frac{\ln x}{x(1+\ln x)^2}\, dx.$$

3405. Points (x, y) are coloured red if $\sin^2 x + \sin^2 y < 1$, and blue if not.

(a) Use a double-angle identity to show that the boundary equation is $\cos 2x + \cos 2y = 0$.

(b) Simplify this to $\cos 2x = \cos(2y + \pi)$.

(c) Hence, verify that the graph of the boundary equation consists, for $m, n \in \mathbb{Z}$, of the lines

$$x + y = \tfrac{\pi}{2} + m\pi,$$
$$x - y = \tfrac{\pi}{2} + n\pi.$$

(d) A point is chosen randomly. Write down the probability that this point is red.

3406. Determine the number of roots of the equation

$$x^{10} - 6x^6 + 9x^2 = 0.$$

3407. The curve $x^2 - y^2 = 1$ is a hyperbola.

(a) Show that, at the points $(\pm 1, 0)$, the tangent to the hyperbola is parallel to the y axis.

(b) By factorising, or otherwise, find the curve's two asymptotes.

(c) Hence, sketch the curve.

3408. State, giving a reason, any conditions on a, x, n which are necessary for this identity to hold:

$$\log_a x^n \equiv n \log_a x.$$

3409. Prove that, if $y = \mathrm{f}(x)$ and $x = \mathrm{f}(y)$ are two non-intersecting polynomial curves, then the shortest path between them must lie along a straight line of the form $x + y = k$.

3410. An expression is given, for $x \in \mathbb{R}$ and $a, b \in \mathbb{N}$, as

$$\lim_{h\to 0} \frac{(x+h)^a - x^a}{(x+h)^b - x^b}.$$

Simplify this expression fully.

3411. A *third-order* differential equation, modelling the position x of a particle at time t, is given by

$$\frac{d^3x}{dt^3} + \frac{d^2x}{dt^2} + \frac{dx}{dt} + x = 0.$$

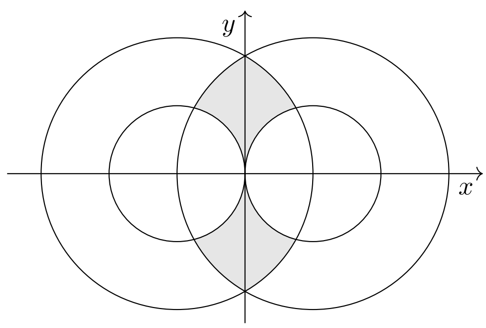
(a) Verify that $x = e^{-t} + \sin t$ is a solution.

(b) Describe this solution's long-term behaviour.

3412. Inequalities defining annular regions are as follows:

$$1 \le (x+1)^2 + y^2 \le 4,$$
$$1 \le (x-1)^2 + y^2 \le 4.$$

The region of the (x, y) plane whose points satisfy both inequalities is shaded in the diagram below:



Find, to 3sf, the area of the shaded region.

3413. The events A and B have probabilities satisfying $\mathbb{P}(A) + \mathbb{P}(B) = 1$ and $\mathbb{P}(A \cap B) = 1/3$. Find

(a) $\mathbb{P}(A' \cap B')$,

(b) $\mathbb{P}(\text{exactly one of } A \text{ or } B)$.

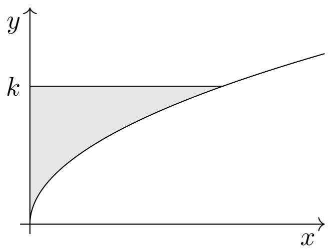
3414. Show that the iteration $x_{n+1} = x_n^2 + 4x_n + 7$ is always increasing, whatever the starting value.

3415. Prove the following identity:

$$2\sin x \sin 2x \equiv \cos x - \cos 3x.$$

3416. Sketch $\log_x a + \log_y a = 0$, for constant $a > 1$.

3417. A region is enclosed by the lines $y = \sqrt{x}$, $x = 0$ and $y = k$, for some positive constant k.



The area of this region is 72. Determine k.

3418. Three coins are tossed. Given that at least two coins have come up tails, find the probability that all three coins have come up tails.

3419. State true or false for each of the following, which concern $\text{hcf}(x, y)$, the highest common factor of natural numbers $x, y > 1$.

(a) $\text{hcf}(x, y) < \min(x, y)$.

(b) $\min(x, y)$ is prime $\implies \text{hcf}(x, y) < \min(x, y)$.

(c) $\max(x, y)$ is prime $\implies \text{hcf}(x, y) < \min(x, y)$.

3420. The graph $y = \text{f}(x)$ is reflected in the line $y = x+k$. Show that the equation of the transformed graph is $x + k = \text{f}(y - k)$.

3421. Prove that, if $y = \text{f}(x)$ is a polynomial graph with a line of symmetry at $x = a$, then $\text{f}'(a) = 0$.

3422. Using the substitution $u^2 = x - 1$, show that

$$\int_5^{17} \frac{1}{(x-1)(1-\sqrt{x-1})}\, dx = \ln \tfrac{4}{9}.$$

3423. A circus gymnast is performing on a vertical pole. At one point, he holds himself horizontally, with his hands shoulder-width apart, 50 cm above one another on the pole. The gymnast is modelled as having mass 60 kg, centred 1.5 metres away from the pole.

(a) Show that his arms each exert a horizontal force of $180g$ Newtons, and determine whether each of these forces is a tension of a thrust.

(b) Explain why it is not possible to determine the vertical forces exerted by each of his hands.

3424. A polynomial $\text{g}(x)$ is decreasing everywhere on $\mathbb{R}$. Show that the range of $\text{g}(x)$ is $\mathbb{R}$.

3425. Prove that the total surface area of a right-circular cone is given, in terms of the semi-vertical angle θ and height h, by

$$A = \pi h^2 \tan\theta(\tan\theta + \sec\theta).$$

3426. Show that $\ln {}^{2k}\text{C}_k - \ln {}^{2k}\text{C}_{k+1} \equiv \ln \frac{k+1}{k}$.

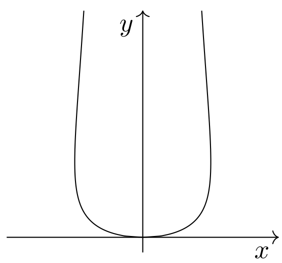
3427. The following function is defined over the largest possible real interval (a, b) of which 0 is an element:

$$\text{f}(x) = \sum_{r=0}^{2} \frac{3}{(x+r)(x+r-1)}$$

Find the range over this domain.

3428. Prove that, if a decimal terminates, then it can be expressed as p/q, where $p, q \in \mathbb{N}$ share no prime factors and the only prime factors of q are 2 and 5.

3429. The diagram below shows part of the curve defined by the implicit relation $x^2(y+1)^2 = y$.



(a) Determine the coordinates of the points at which the tangent is parallel to $\mathbf{j}$.

(b) Show that, as $y \to \infty$, both branches of the curve tend asymptotically to the y axis.

3430. A particle has position $\mathbf{r} = \begin{pmatrix} \sin t \\ \cos 2t \\ \sin 2t \end{pmatrix}$ m.

Find the first time $t > 0$ at which the speed of the particle is 2 ms^{-1}.

3431. The set of values for which the polynomial graph $y = \text{f}(x)$ is convex is $x \in (\infty, a) \cup (b, \infty)$, where $a < b$. Prove that f has degree at least 4.

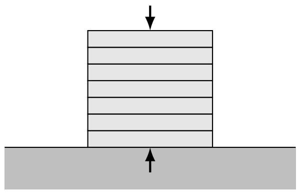
3432. Shade the region of the (x, y) plane which satisfies both of the following inequalities:

$$(x-a)(y-b) \geq 0,$$
$$(x-a)^2 + (y-b)^2 < 1.$$

3433. Solve the equation $\tan 2a - \cot a = 0$ for $a \in [0, 2\pi)$

3434. A GP has first three terms $100, a, b$, and an AP has first three terms $100, a(b-1), a$. Find a and b.

3435. In a warehouse, n boxes, each of mass 2 kg, are bound together in a vertical stack, which stands on flat ground. The binding exerts a pair of forces, as depicted below, upwards on the lowest box and downwards on the highest, of magnitude 50 N.



(a) Determine the magnitude of the contact force between the topmost and second boxes.

(b) Find an expression for the total contact force (from the ground and binding combined) on the base of the bottom box.

(c) There is a possibility of a box buckling if it experiences a contact force of more than 500 Newtons on any one of its surfaces. Determine the greatest number of boxes that can safely be bound in such a stack.

3436. One of the following statements is true; the other is not. Identify and disprove the false statement.

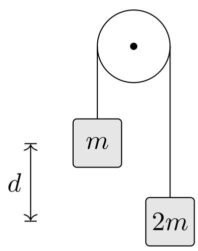
① $x^5 + x^3 - x = 0 \implies x = 0$,
② $x^5 + x^3 + x = 0 \implies x = 0$.

3437. Describe the transformation that takes the graph $x = (y-p)^2 + q$ onto the graph $x = (y-2p)^2 + q$.

3438. Prove the integration by parts formula

$$\int u \frac{dv}{dx}\, dx = uv - \int \frac{du}{dx} v\, dx.$$

3439. Two loads, with masses m and $2m$ kg, are attached by a light, inextensible string which is passed over a smooth pulley. They are released from rest, at $t = 0$, at the same height. After t seconds, the vertical distance between their centres is d metres.



Find d in terms of t.

3440. Solve $\log_2(4 \times 8^x) - \log_4(2 \times 16^x) = 1$.

3441. State, giving a reason, which of the implications $\implies$, $\impliedby$, $\iff$ links the following statements regarding a polynomial function f:

① f is increasing on $\mathbb{R}$,
② f is invertible on $\mathbb{R}$.

3442. For the circle $x^2 + y^2 = 1$, differentiate implicitly to show that

$$\frac{d^2y}{dx^2} = \frac{x^2 - y^2}{y^3}.$$

3443. Prove that, if two cubics $y = \mathrm{f}(x)$ and $y = \mathrm{g}(x)$ cross each other exactly twice, then they have the same leading coefficient.

3444. In this question, f is a generic function, and g and h are specific functions defined by

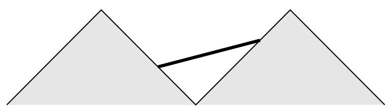
$$\mathrm{g} : x \mapsto 2x,$$
$$\mathrm{h} : x \mapsto x + 3.$$

Describe the transformations that map the graph $y = \mathrm{f}(x)$ to

(a) $y = \mathrm{ghf}(x)$,
(b) $y = \mathrm{fhg}(x)$,
(c) $\mathrm{gh}(y) = \mathrm{ghf}(x)$.

3445. Divide $x^4 - x^3 + 16x^2 - 6x + 80$ by $x^2 + x + 8$.

3446. A roofer places a rigid board between two adjacent roofs, as shown below. Both roofs are rough, and they are perpendicular to one another. The board rests in equilibrium.



State, with justification, the direction in which the friction acts at each end of the board.

3447. A function is defined, for $k \in \mathbb{N}$, by

$$\mathrm{h}(x) = x^{2k+1} - x^{2k+3}.$$

Show carefully that $y = \mathrm{h}(x)$ is inflected at $x = 0$.

3448. Prove that $\tan x \equiv \dfrac{\sin 2x}{\cos 2x + 1}$.

3449. Find $\mathbb{P}(X^2 - X < 1)$, where $X \sim N(0, 1)$.

3450. Prove by contradiction that, if a polynomial graph $y = \mathrm{f}(x)$ is convex for all $x \in \mathbb{R}$, then no three points on it are collinear.

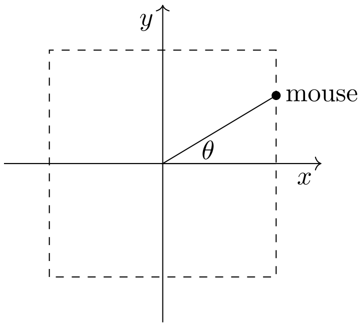
3451. Prove the following integral, for constants a, b:

$$\int \frac{a+x}{b+x}\, dx = (a-b) \ln|b+x| + x + c.$$

3452. Show, by eliminating θ or otherwise, that these are the parametric equations of a circle which passes through the origin:

$$x = 0.6 + \cos\left(\theta + \tfrac{\pi}{6}\right),$$
$$y = 0.8 - \sin\left(\theta + \tfrac{\pi}{6}\right).$$

3453. A mouse runs at a constant speed of 1 ms^{-1} around the perimeter of a square of side length 1 m, whose centre is at the origin. The direction of the mouse's position vector is defined as θ. At $t = 0$, the mouse is on the positive x axis, and θ increases with t.



(a) Show that, before the mouse reaches the first corner, $\tan\theta = 2t$.

(b) Determine upper and lower bounds, over the whole motion, on the rate of change of θ. Give your answers in radians per second.

3454. The graph $y = x^3 + x$ can be transformed to the graph $y = x^3 + 6x^2 + 13x$ by a translation. Find the translation vector.

3455. Evaluate $\displaystyle\lim_{x\to\ln 2} \frac{e^{2x} - 4}{e^x - 2}$.

3456. A function f is given, for constant $k \in \mathbb{R}$, as

$$\mathrm{f}(x) = \sin x - k^3 \tan x.$$

By considering stationary points of $y = \mathrm{f}(x)$, show that, if $k \notin (0, 1]$, then f is invertible on the domain $(-\pi/2, \pi/2)$.

3457. The straight line through (e^c, e^{-c}) and $(-2, e^c)$ has gradient $-3/8$. Find c.

3458. Sketch $\sqrt{y} = x^3 - x$.

3459. From a large jar filled with equal numbers of black and white counters, two counters are chosen and placed in a bag. Then, a black counter is added to the bag. The bag is shaken, and one counter is taken out. This counter turns out to be black. Find the probability that the remaining counters are both black.

3460. Prove that, if a prime number can be expressed as the difference of two nth powers $a^n - b^n$, where $n \in \{2, 3, ...\}$, and $a, b \in \mathbb{N}$, then $a = b + 1$.

3461. State, with a reason, whether $y = x^4$ intersects the following curves:

(a) $y = x^2 + 1$,

(b) $y = x^4 + 1$,

(c) $y = x^6 + 1$.

3462. An arithmetic sequence U_n begins $\sin\theta, \cos\theta, ...$ Show that the range of U_5 is $[-5, 5]$.

3463. A firework is built to explode into two projectile pieces, moving in opposite directions with initial speed u. Prove that, in the subsequent motion, the distance between the pieces is $d = 2ut$.

3464. Set S consists of all lines of the form $y = mx + c$ which pass through the point (a, b). Find the DE whose solution set is S, giving your answer in the form

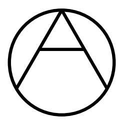
$$\frac{dy}{dx} = \frac{\mathrm{f}(y)}{\mathrm{g}(x)}.$$

3465. Prove that there is no number that appears in both of the following sequences, defined for $n \in \mathbb{N}$:

$$A_n = 2 \times 3^n,$$
$$B_n = 4 \times 5^n.$$

3466. Either prove or disprove the following statement: "A set of n linear equations in n unknowns always has a unique solution point $(x_1, x_2, ..., x_n)$."

3467. A logo consists of the letter A inside a circle. The five line segments of the letter A all have length 1.



Determine the area of the circle.

3468. Show that the closed curve $x^4 + y^4 = 1$ and the line $y = x + 1$ intersect exactly twice.

3469. A variable X has distribution $B(n, 1/4)$. It is given that, for some value of n,

$$\mathbb{P}(X = 2) = \mathbb{P}(X = 3).$$

Using an algebraic method, determine n.

3470. Two curves are defined as

$$x^2 y + y^2 = 1,$$
$$x^2 + y^2 = 1.$$

Show that these curves are tangent at the y axis.

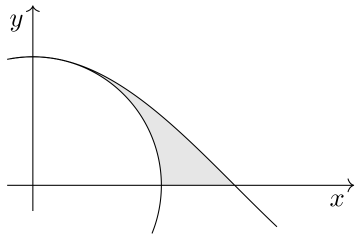
3471. Two particles move with position vectors given, at time $t \in \mathbb{R}$, by

$$\mathbf{r}_1 = (2t-7)\mathbf{i} + 2\mathbf{j} + (3t-2)\mathbf{k},$$
$$\mathbf{r}_2 = t\mathbf{i} + (1-t)\mathbf{j} + (4-3t)\mathbf{k}.$$

(a) Show that the particles do not collide.
(b) Show that the paths of the particles intersect.

3472. Solve the equation $e^{2x} + 12e^{3x} = e^x$.

3473. The curves below are $y = \cos x$ and $x^2 + y^2 = 1$.



Find the exact area of the shaded region.

3474. Show that no value of x satisfies

$$5 - 3|x-3| > x^2 - 1.$$

3475. By proposing $p\sqrt{2} + q\sqrt{5}$, determine the square root of $47 + 6\sqrt{10}$.

3476. A virologist is modelling the spread of a virus through a population of primates. The prevalence of the disease in a given region is denoted by the unitless variable P. The virologist takes the rate of change of prevalence as being proportional to the prevalence and to a linear function of time t.

(a) Solve a differential equation in P and t to show that prevalence may be expressed as

$$P = P_0 e^{at + \frac{1}{2}bt^2}.$$

(b) State conditions on the constants a and b such that the model predicts an initial increase in prevalence, but does not predict $P \to \infty$.
(c) Show that, if these conditions are fulfilled, then prevalence peaks at $t = -\frac{a}{b}$.

3477. Disprove the following claim: "If $f'(x)$ has a factor of $(x-\alpha)$, then $f(x)$ has a factor of $(x-\alpha)^2$."

3478. Determine the coordinates of any local maxima of the curve $y = x^4 e^{-x}$. Give an exact answer.

3479. Using the substitution $z = y/x$, or otherwise, show that the locus of the following relationship is a pair of perpendicular lines:

$$\frac{y}{x} - \frac{x}{y} = 1.$$

3480. The sequences a_n and b_n, which are arithmetic and geometric respectively, are both divergent. Their terms satisfy

$$a_1 = b_1,$$
$$a_3 = b_3,$$
$$a_5 = b_4 - b_1.$$

Show that $b_{n+1} = 2b_n$.

3481. Show that $\displaystyle\int_0^{n\pi} |\sin x|\, dx = 2n$, for $n \in \mathbb{N}$.

3482. A generic parabola has equation

$$y = ax^2 + bx + c.$$

Prove that, if tangents are drawn to the parabola at $x = \pm q$, then they meet on the y axis.

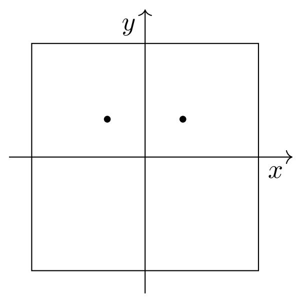
3483. Sketch the graph $y = \ln(x+1)\cos x$, for $x \in [0, \infty)$. You don't need to find intercepts or SPs.

3484. The *largest empty circle* problem involves finding the largest circle in a given space containing none of a particular set of points.

Let the space in question be the square

$$\{(x, y) : -1 \le x \le 1, -1 \le y \le 1\}.$$

Determine the radius of the largest empty circle containing neither of the points $(\pm 1/3, 1/3)$.



3485. Show that $\dfrac{d^2}{dx^2}\left(\dfrac{\sin x}{2 + \cos x}\right) = \dfrac{2\sin x(\cos x - 1)}{(2 + \cos x)^3}$.

3486. By considering the signs of the factors, shade the region(s) of the (x, y) plane that satisfy

$$(x+y)(x-y) < 0.$$

3487. Show that $\operatorname{cosec}\theta$ can be expressed as $\pm f(\sec\theta)$, where the function f is to be determined. It should involve no inverse trigonometric functions.

3488. The letters of the word ABSTRACT are arranged in a random order. Find the probability that the rearrangement ends with AA.

3489. Solve the equation $\tan|x| = \sqrt{3}$, for $x \in [-\pi, \pi]$.

3490. In this question, do not use a calculator.

You are given that $f(x) = 6x^3 + 29x^2 + 46x + 24$ can be expressed as $(ax+b)(bx+c)(cx+d)$, where a, b, c, d are distinct natural numbers. Determine the roots of f.

3491. At a point $\big(x, f(x)\big)$, the *radius of curvature* of a graph $y = f(x)$ is the radius R of the circular arc which best approximates $f(x)$.

(a) For $f(x) = x^2$, find $f''(0)$.
(b) For $g(x) = R - \sqrt{R^2 - x^2}$, find $g''(0)$.
(c) Determine R such that $f''(0) = g''(0)$.
(d) On a single set of axes, sketch $y = f(x)$ and $y = g(x)$.

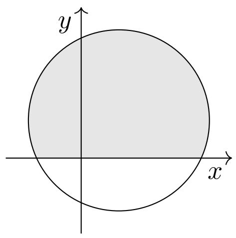
3492. A differential equation is given as

$$\left(\frac{dy}{dx}\right)^2 + y = x.$$

Show that no cubic $y = f(x)$ satisfies this.

3493. The circle shown below has equation

$$(2x-1)^2 + (2y-1)^2 = 10.$$



Find the area of the shaded region, to 3sf.

3494. A student is discussing a numerical approximation to the definite integral

$$\int_{0.25}^{0.75} e^x \sin\left(x^2\right) dx.$$

The student says: "The trapezium rule will give an overestimate for the integral because f is increasing on the domain $[0.25, 0.75]$".

Explain the error and correct it.

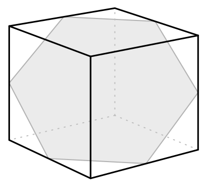
3495. A pentagon has the following properties:

① a line of reflective symmetry,
② no reflex interior angles,
③ three sides of length 1,
④ two sides of length 2, with an interior angle of 60° at their shared vertex.

Find the area of the pentagon.

3496. The function $h(x) = \cos ax + \cos bx$ is defined over the real numbers, where $a, b \in \mathbb{N}$ are odd numbers. Determine the range of h.

3497. The diagram shows a cube of unit side length, and a regular hexagon joining the midpoints of six of the cube's edges.



Determine the acute angle between the hexagon and a face of the cube. (The angle between planes is defined as the angle between their normals.)

3498. Take $g = 10$ in this question.

A projectile is launched horizontally from the point $(0, 1)$ with speed $\sqrt{10}$ ms^{-1}. It bounces, losing no speed, on the ground at $y = 0$.

(a) Find the equation of the trajectory before the first bounce.
(b) Show that bounces are $2\sqrt{2}$ m apart.
(c) Hence, or otherwise, show that the equation of the trajectory after the first bounce is

$$y = -3 + 2\sqrt{2}x - \tfrac{1}{2}x^2.$$

3499. A curve is given by $y = \dfrac{\cos x}{1 - \sin x}$.

(a) Find and simplify $\dfrac{dy}{dx}$.
(b) Determine the equation of the normal to the curve at $x = 0$.
(c) This normal meets the curve again between $x = 2$ and $x = 3$. Using a numerical method, find the intersection to 4 significant figures.

3500. Two spheres in (x, y, z) space are described by

$$x^2 + y^2 + z^2 = 1,$$
$$(x-3)^2 + (y-4)^2 + z^2 = 1.$$

Find the shortest distance between the spheres.

——— End of 35th Hundred ———